

On Previdi's delooping conjecture for K -theory

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Abstract

We prove, under a mild assumption, Previdi's conjecture stating that the Waldhausen space (K-theory space) of an exact category \mathcal{A} admits a delooping by the Waldhausen space (K-theory space) of Beilinson's category $\varprojlim \mathcal{A}$ of locally compact objects over \mathcal{A} .

1 Introduction

Let \mathcal{A} be an exact category. We define the *Waldhausen space* $S(\mathcal{A})$ of the exact category \mathcal{A} to be the geometric realization of the simplicial category $iS_\bullet(\mathcal{A})$, where $iS_n(\mathcal{A})$ is the subcategory of isomorphisms of the n -th Waldhausen S_\bullet -construction. The loop space $K(\mathcal{A}) = \Omega S(\mathcal{A})$ is the algebraic K -theory space of \mathcal{A} ([12]). In his thesis [8], Previdi conjectured the following delooping statement.

Conjecture 1.1 (Previdi [8], 5.1.7) *The space $S(\varprojlim \mathcal{A})$ is a delooping of $S(\mathcal{A})$.*

Here $\varprojlim \mathcal{A}$ is the category introduced by Beilinson in [3], A.3. He stated the conjecture assuming \mathcal{A} to be partially abelian. The exact category \mathcal{A} is said to be *partially abelian* if \mathcal{A} and its opposite have pullbacks of admissible monomorphisms with common target. See [8], 3.2.7, 3.2.9, for details.

We prove the conjecture under another mild assumption on \mathcal{A} .

Theorem 1.2 *There is a homotopy equivalence $K(\mathcal{A}) \xrightarrow{\sim} \Omega K(\varprojlim \mathcal{A})$ if \mathcal{A} is idempotent complete.*

(Although we have replaced the Waldhausen spaces by the K -theory spaces, Previdi's original statement turns out to be true under the same condition, with a slightly different proof. See section 3.1.) Recall that an exact category is said to be *idempotent complete* if every idempotent has a kernel. We note that this is the case for most of the typical examples such as the category of finitely generated projective modules over a ring, the category of locally free sheaves of finite rank on a scheme, and any abelian category. Moreover, for any exact category \mathcal{A} there is an idempotent complete exact category $\tilde{\mathcal{A}}$, called the idempotent completion, such that there is a cofinal embedding $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ ([11], A.9.1). The cofinality of $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ means that the induced map $K_i(\mathcal{A}) \rightarrow K_i(\tilde{\mathcal{A}})$ of K -groups is an isomorphism for $i > 0$, and a monomorphism for $i = 0$. Therefore our assumption is not at all restrictive in the K -theoretical context.

Before going to the proof, let us briefly review the background of Previdi's conjecture. Previdi introduced a categorical generalization of Kapranov's work in [6] on determinantal anomaly for

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Tate spaces. A Tate space is by definition a locally linearly compact vector space. In other words, it is a “locally compact object” over the exact category $\mathbf{Vect}_0(k)$ of finite dimensional vector spaces. Previdi [9] showed that for the general exact category \mathcal{A} , the notion of local compactness can be defined by using Beilinson’s category $\varprojlim \mathcal{A}$. In this sense an object of $\varprojlim \mathcal{A}$ is called a *generalized Tate space*.

An advantage of using Beilinson’s category is that the construction \varprojlim can be iterated since $\varprojlim \mathcal{A}$ is exact. In particular we get the notion of an *n-Tate space* as an object of the *n*-times iteration $\varprojlim^n \mathbf{Vect}_0(k)$. (This motivational example is also idempotent complete, since $\mathbf{Vect}_0(k)$ is abelian and \varprojlim preserves idempotent completeness, by Proposition 2.3 below.) The concept of higher dimensional (especially 2 dimensional) Tate spaces is important in the representation theory of double loop groups. See [1] and [4], for instance.

Previdi [8] extended Kapranov’s construction of the dimensional torsor and determinantal gerbe by defining Sato Grassmannians for generalized Tate spaces, proving the important property called the *multiplicativity*. The key observation here is that the multiplicativity of the dimensional torsor, which is defined as a $K_0(\mathcal{A})$ -torsor, can be described as a data on 2-simplices of the S_\bullet -construction of $\varprojlim \mathcal{A}$. Similarly, the multiplicativity of the determinantal $K_1(\mathcal{A})$ -gerbe is interpreted as a data on 3-simplices of $iS_\bullet(\varprojlim \mathcal{A})$. These data define a cohomology class in $H^{i+2}(S(\varprojlim \mathcal{A}), K_i(\mathcal{A}))$, and hence a map $H_{i+2}(S(\varprojlim \mathcal{A})) \rightarrow K_i(\mathcal{A})$, for $i = 0, 1$. Composing with the Hurewicz homomorphism $K_{i+1}(\varprojlim \mathcal{A}) = \pi_{i+2}(S(\varprojlim \mathcal{A})) \rightarrow H_{i+2}(S(\varprojlim \mathcal{A}))$, one sees that there is a canonically defined map $K_{i+1}(\varprojlim \mathcal{A}) \rightarrow K_i(\mathcal{A})$, for $i = 0, 1$. Previdi expected that these are low dimensional instances of Conjecture 1.1.

In [8], Previdi suggested that his theory on the dimensional torsor and determinantal gerbe can be generalized to higher dimensions, as the “homotopy dimensional torsor” that would induce all the higher dimensional analogues of the maps $K_{i+1}(\varprojlim \mathcal{A}) \rightarrow K_i(\mathcal{A})$, $i = 0, 1$, above constructed. On the other hand, we have an isomorphism $K_i(\mathcal{A}) \rightarrow K_{i+1}(\varprojlim \mathcal{A})$, for all $i \geq 0$, resulting from the homotopy equivalence in Theorem 1.2.

Problem 1.3 Compare the two maps $K_{i+1}(\varprojlim \mathcal{A}) \rightarrow K_i(\mathcal{A})$ and $K_i(\mathcal{A}) \rightarrow K_{i+1}(\varprojlim \mathcal{A})$, for $i = 0, 1$.

Our hope is that they are inverse to each other. If this is correct, the “homotopy dimensional torsor” would be characterized as giving rise to an inverse to our isomorphism $K_i(\mathcal{A}) \rightarrow K_{i+1}(\varprojlim \mathcal{A})$.

Previdi’s conjecture, now our theorem, is interesting as it gives another answer to a problem posed in the algebraic K-theoretical context, from the viewpoint of the different direction of research concerning Tate spaces. It was part of Problem 5 of the Lake Louise Problem Session [5] to give a functor which assigns to an exact category \mathcal{A} an exact category \mathcal{SA} such that $K(\mathcal{SA})$ is a delooping of $K(\mathcal{A})$. This problem was first solved by Schlichting [10], by defining \mathcal{SA} as a localization of the category of countably indexed admissible ind-objects in \mathcal{A} .

We recall the definition and properties of Beilinson’s category $\varprojlim \mathcal{A}$ following Beilinson [3] and Previdi [9] in section 2, and prove Theorem 1.2 using results of Schlichting [10] in section 3. We follow the notations adopted in [8] and [9]. For instance we write $\mathrm{Ind}_{\mathbb{N}}^a \mathcal{A}$ for what is denoted by \mathcal{FA} in [10], and $\mathrm{Fun}^a(\Pi, \mathcal{A})$ instead of \mathbf{A}_a^Π which is used in [3].

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2 Beilinson's category $\varprojlim \mathcal{A}$

2.1 Ind- and pro-objects in a category

We first recall some generalities on ind- and pro-objects. For any category \mathcal{C} , the category $\text{Ind } \mathcal{C}$ (resp. $\text{Pro } \mathcal{C}$) of *ind-objects* (resp. *pro-objects*) in \mathcal{C} is defined to have as objects functors $\mathcal{X} : J \rightarrow \mathcal{C}$ with domain J small and filtering (resp. $\mathcal{X} : I^{\text{op}} \rightarrow \mathcal{C}$ with I small and filtering). The ind-object $\mathcal{X} : J \rightarrow \mathcal{C}$ (resp. pro-object $\mathcal{X} : I^{\text{op}} \rightarrow \mathcal{C}$) defines a functor $\mathcal{C}^{\text{op}} \rightarrow (\text{sets})$, $C \mapsto \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(C, \mathcal{X}_j)$ (resp. $\mathcal{C} \rightarrow (\text{sets})$, $C \mapsto \varprojlim_{i \in I} \text{Hom}_{\mathcal{C}}(\mathcal{X}_i, C)$). A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of ind-objects (resp. pro-objects) is a natural transformation between the functors $\mathcal{C}^{\text{op}} \rightarrow (\text{sets})$ (resp. $\mathcal{C} \rightarrow (\text{sets})$) associated to \mathcal{X} and \mathcal{Y} . If \mathcal{X} and \mathcal{Y} have a common index category, a natural transformation $\mathcal{X} \rightarrow \mathcal{Y}$ between the functors \mathcal{X} and \mathcal{Y} defines a map between the ind- or pro-objects \mathcal{X} and \mathcal{Y} . Conversely, every map of ind- or pro-objects $\mathcal{X} \rightarrow \mathcal{Y}$ can be “straightified” into a natural transformation, in the sense that there is a commutative diagram in $\text{Ind } \mathcal{C}$ or $\text{Pro } \mathcal{C}$

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \sim \downarrow & & \sim \downarrow \\ \tilde{\mathcal{X}} & \longrightarrow & \tilde{\mathcal{Y}} \end{array}$$

with the vertical maps isomorphisms, $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ having a common index category, and $\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ coming from a natural transformation. (See [2], Appendix, for details.)

If \mathcal{C} is an exact category, the categories $\text{Ind } \mathcal{C}$ and $\text{Pro } \mathcal{C}$ possess exact structures. A pair of composable morphisms in $\text{Ind } \mathcal{C}$ or $\text{Pro } \mathcal{C}$ is an admissible short exact sequence if it can be straightified into a sequence of natural transformations which is level-wise exact in \mathcal{C} ([9], 4.15, 4.16). We also remark that in this case the functor $C \mapsto \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(C, \mathcal{X}_j)$ associated to the ind-object \mathcal{X} can be considered to take values in the category of abelian groups. This defines a fully exact embedding of $\text{Ind } \mathcal{C}$ into the *abelian envelope* $\text{Lex } \mathcal{C}$ of \mathcal{C} , that is the abelian category of left exact additive functors $\mathcal{C}^{\text{op}} \rightarrow (\text{abelian groups})$. (Here we are calling the embedding *fully exact* to mean that the image of $\text{Ind } \mathcal{C}$ is closed under extensions in $\text{Lex } \mathcal{C}$, and a pair of composable maps in $\text{Ind } \mathcal{C}$ is an admissible short exact sequence if and only if its image is a one in $\text{Lex } \mathcal{C}$.)

2.2 Definition of $\varprojlim \mathcal{A}$, and the categories $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$

Let \mathcal{A} be an exact category. We write Π for the ordered set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq j\}$, where $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. A functor $X : \Pi \rightarrow \mathcal{A}$, where Π is viewed as a filtered category, is

admissible if for every triple $i \leq j \leq k$, the sequence $X_{i,j} \hookrightarrow X_{i,k} \twoheadrightarrow X_{j,k}$ is an admissible short exact sequence in \mathcal{A} . We denote by $\text{Fun}^a(\Pi, \mathcal{A})$ the exact category of admissible functors $X : \Pi \rightarrow \mathcal{A}$ and natural transformations, where a sequence $X \rightarrow Y \rightarrow Z$ of admissible functors is an admissible short exact sequence in $\text{Fun}^a(\Pi, \mathcal{A})$ if $X_{i,j} \hookrightarrow Y_{i,j} \twoheadrightarrow Z_{i,j}$ is an admissible short exact sequence in \mathcal{A} for every $i \leq j$. A bicofinal map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ (ϕ is said to be *bicofinal* if it is nondecreasing and satisfies $\lim_{i \rightarrow \pm\infty} \phi(i) = \pm\infty$) induces a cofinal functor $\tilde{\phi} : \Pi \rightarrow \Pi$, $(i, j) \mapsto (\phi(i), \phi(j))$. If ϕ and $\psi : \mathbb{Z} \rightarrow \mathbb{Z}$ are bicofinal maps such that $\phi(i) \leq \psi(i)$ for all i , and if $X : \Pi \rightarrow \mathcal{A}$ is an admissible functor, there is a natural transformation $u_{X,\phi,\psi} : X \circ \tilde{\phi} \rightarrow X \circ \tilde{\psi}$.

Definition 2.1 (Beilinson [3], A.3) *The category $\varprojlim \mathcal{A}$ is defined to be the localization of $\text{Fun}^a(\Pi, \mathcal{A})$ by the morphisms $u_{X,\phi,\psi}$, where $X \in \text{ob } \text{Fun}^a(\Pi, \mathcal{A})$, and $\phi \leq \psi : \mathbb{Z} \rightarrow \mathbb{Z}$ are bicofinal.*

If $X : \Pi \rightarrow \mathcal{A}$ is an admissible functor, we have for each $j \in \mathbb{Z}$ a pro-object $X_{\bullet,j} : \{i \in \mathbb{Z} \mid i \leq j\} \rightarrow \mathcal{A}$, $i \mapsto X_{i,j}$, in \mathcal{A} . We get in turn an ind-object $\mathbb{Z} \rightarrow \text{Pro } \mathcal{A}$, $j \mapsto X_{\bullet,j}$, in $\text{Pro } \mathcal{A}$. Thus the admissible functor X can be viewed as an object of the iterated Ind-Pro category $\text{Ind Pro } \mathcal{A}$. If $\phi \leq \psi : \mathbb{Z} \rightarrow \mathbb{Z}$ are bicofinal, the map $u_{X,\phi,\psi}$ defines an isomorphism between the ind-pro-objects $X \circ \tilde{\phi}$ and $X \circ \tilde{\psi}$. We get a functor $\varprojlim \mathcal{A} \rightarrow \text{Ind Pro } \mathcal{A}$. In view of the following theorem, we regard $\varprojlim \mathcal{A}$ as an exact subcategory of $\text{Ind Pro } \mathcal{A}$.

Theorem 2.2 (Previdi [9], 5.8, 6.1) *The functor $\varprojlim \mathcal{A} \rightarrow \text{Ind Pro } \mathcal{A}$ is fully faithful. Moreover, the image is closed under extensions in $\text{Ind Pro } \mathcal{A}$. In particular, $\varprojlim \mathcal{A}$ has an exact structure where a sequence in $\varprojlim \mathcal{A}$ is exact if and only if its image in $\text{Ind Pro } \mathcal{A}$ is exact.*

Previdi also showed that the categories $\text{Ind}_{\mathbb{N}_0}^a \mathcal{A}$ of countably indexed ind-objects with structure maps admissible monomorphisms, and $\text{Pro}_{\mathbb{N}_0}^a \mathcal{A}$ of countably indexed pro-objects with structure maps admissible epimorphisms, are exact ([9], 6.3). This is done by constructing fully faithful embeddings of $\text{Ind}_{\mathbb{N}_0}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}_0}^a \mathcal{A}$ into $\varprojlim \mathcal{A}$ which are closed under extensions.

Here we use, instead of $\text{Ind}_{\mathbb{N}_0}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}_0}^a \mathcal{A}$, equivalent categories $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$. An object of $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ (resp. $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$) is a functor $X : \mathbb{N} \rightarrow \mathcal{A}$ (resp. $X : \mathbb{N}^{\text{op}} \rightarrow \mathcal{A}$), where \mathbb{N} is the set of natural numbers viewed as a filtered category by the standard order, such that for every $j \leq j'$ the map $X_j \hookrightarrow X_{j'}$ is an admissible monomorphism in \mathcal{A} (resp. for every $i \leq i'$ the map $X_i \twoheadleftarrow X_{i'}$ is an admissible epimorphism in \mathcal{A}). The embedding $\text{Ind}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \varprojlim \mathcal{A}$ (resp. $\text{Pro}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \varprojlim \mathcal{A}$) is given by sending $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \dots \in \text{ob } \text{Ind}_{\mathbb{N}}^a \mathcal{A}$ (resp. $X_1 \twoheadleftarrow X_2 \twoheadleftarrow X_3 \twoheadleftarrow \dots \in \text{Pro}_{\mathbb{N}}^a \mathcal{A}$) to the object in $\varprojlim \mathcal{A}$ determined by $X_{i,j} = X_{0,j} = X_j$ for $i \leq 0 < j$ (resp. $X_{i,j} = X_{i,1} = X_{-i+1}$ for $i \leq 0 < j$). There are also embeddings of \mathcal{A} into $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ (hence into $\varprojlim \mathcal{A}$) that sends $A \in \text{ob } \mathcal{A}$ to the ind- or pro-object $A = A = A = \dots$.

Proposition 2.3 *If \mathcal{A} is idempotent complete, then so are $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$, $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$, and $\varprojlim \mathcal{A}$.*

(Proof) Let $p : X \rightarrow X$ be an idempotent in $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$, $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$, or $\varprojlim \mathcal{A}$. By the straightification argument we may assume that p is given as a natural transformation which consists of an idempotent $p_\alpha : X_\alpha \rightarrow X_\alpha$ on each X_α , where $\alpha \in \mathbb{N}$ or Π . Since \mathcal{A} is idempotent complete, each p_α has a kernel $\ker p_\alpha$ in \mathcal{A} and the structure maps $X_\alpha \rightarrow X_\beta$ give rise to maps $\ker p_\alpha \rightarrow \ker p_\beta$. To see that p has a kernel in $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$, $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$, or $\varprojlim \mathcal{A}$, it suffices to check that this diagram consisting

of $\ker p_\alpha$ is admissible. For $\varinjlim \mathcal{A}$, for instance, this amounts to show that for every $i \leq j \leq k$ the sequence $\ker p_{i,j} \rightarrow \ker p_{i,k} \xrightarrow{\hookrightarrow} \ker p_{j,k}$ is an admissible short exact sequence in \mathcal{A} . Since \mathcal{A} is a fully exact subcategory of an abelian category (e.g. $\text{Lex } \mathcal{A}$), it is enough to show that the sequence is a short exact sequence in that abelian category. Then we may proceed as in the category of abelian groups, i.e. may think of “elements” and use diagram-chase techniques. (See [7], VIII-4, Theorem 3.) We have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker p_{i,j} & \longrightarrow & \ker p_{i,k} & \longrightarrow & \ker p_{j,k} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_{i,j} & \xrightarrow{m} & X_{i,k} & \xrightarrow{e} & X_{j,k} \longrightarrow 0 \\
& & \downarrow p_{i,j} & & \downarrow p_{i,k} & & \downarrow p_{j,k} \\
0 & \longrightarrow & X_{i,j} & \xrightarrow{m} & X_{i,k} & \xrightarrow{e} & X_{j,k} \longrightarrow 0,
\end{array}$$

where the middle and bottom horizontal sequences are exact. It is part of the snake lemma that the top horizontal sequence is exact at $\ker p_{i,j}$ and $\ker p_{i,k}$. To show that it is exact at $\ker p_{j,k}$, take $z \in \ker p_{j,k} \subset X_{j,k}$ and fix $y \in X_{i,k}$ such that $e(y) = z$. We have $p_{i,k}(y - p_{i,k}(y)) = 0$ as $p_{i,k}$ is an idempotent, and $e(y - p_{i,k}(y)) = z - p_{j,k}(z) = z$. Hence $\ker p_{i,k} \rightarrow \ker p_{j,k}$ is surjective, and we get the desired conclusion for $\varinjlim \mathcal{A}$. The proofs for $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ and $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ are similar. \blacksquare

We refer to [9] for detailed discussions on ind/pro-objects in exact categories.

3 Proof of Theorem 1.2

We prove the theorem using the s -filtering localization sequence constructed by Schlichting [10].

Let $\mathcal{A} \hookrightarrow \mathcal{U}$ be a fully faithful embedding of exact categories. Following Schlichting [10], we define a map in \mathcal{U} to be a *weak isomorphism* with respect to $\mathcal{A} \hookrightarrow \mathcal{U}$ if it is either an admissible monomorphism that admits a cokernel in the image of $\mathcal{A} \hookrightarrow \mathcal{U}$ or an admissible epimorphism that admits a kernel in the image of $\mathcal{A} \hookrightarrow \mathcal{U}$. In particular, for every $A \in \text{ob } \mathcal{A}$ the maps $0 \rightarrow A$ and $A \rightarrow 0$ are weak isomorphisms. The localization of \mathcal{U} by weak isomorphisms with respect to \mathcal{A} is denoted by \mathcal{U}/\mathcal{A} . Recall, from [10], that the fully faithful embedding $\mathcal{A} \hookrightarrow \mathcal{U}$ of exact categories is a *left s -filtering* if the following conditions are satisfied.

- (1) If $A \twoheadrightarrow U$ is an admissible epimorphism in \mathcal{U} with $A \in \text{ob } \mathcal{A}$, then $U \in \text{ob } \mathcal{A}$.
- (2) If $U \hookrightarrow A$ is an admissible monomorphism in \mathcal{U} with $A \in \text{ob } \mathcal{A}$, then $U \in \text{ob } \mathcal{A}$.
- (3) Every map $A \rightarrow U$ in \mathcal{U} with $A \in \text{ob } \mathcal{A}$ factors through an object $B \in \text{ob } \mathcal{A}$ such that $B \hookrightarrow U$ is an admissible monomorphism in \mathcal{U} .
- (4) If $U \twoheadrightarrow A$ is an admissible epimorphism in \mathcal{U} with $A \in \text{ob } \mathcal{A}$, then there is an admissible monomorphism $B \hookrightarrow U$ with $B \in \text{ob } \mathcal{A}$ such that the composition $B \twoheadrightarrow A$ is an admissible epimorphism in \mathcal{A} .

A *right s -filtering* embedding is defined by dualizing the conditions above.

Theorem 3.1 (Schlichting [10], 1.16, 2.1) *If the embedding $\mathcal{A} \hookrightarrow \mathcal{U}$ is left or right s -filtering, then the localization \mathcal{U}/\mathcal{A} has an exact structure where a short sequence is exact if and only if it is isomorphic to the image of an admissible short exact sequence in \mathcal{U} . Moreover, if \mathcal{A} is idempotent complete, the sequence of exact categories $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ induces a homotopy fibration $K(\mathcal{A}) \rightarrow K(\mathcal{U}) \rightarrow K(\mathcal{U}/\mathcal{A})$ of K -theory spaces.*

Lemma 3.2 (Schlichting [10], 3.2, 3.4) *If \mathcal{A} is idempotent complete, then the embedding $\mathcal{A} \hookrightarrow \text{Ind}_{\mathbb{N}}^a \mathcal{A}$ is left s -filtering, and the K -theory space $K(\text{Ind}_{\mathbb{N}}^a \mathcal{A})$ is contractible. In particular, there is a homotopy equivalence between $K(\mathcal{A})$ and $\Omega K(\text{Ind}_{\mathbb{N}}^a \mathcal{A}/\mathcal{A})$.*

Remember the embedding $\text{Pro}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \varprojlim \mathcal{A}$, that sends an object $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ to the object of $\varprojlim \mathcal{A}$ determined by $X_{i,j} = X_{i,1} = X_{-i+1}$ ($i \leq 0 < j$).

Lemma 3.3 *If \mathcal{A} is idempotent complete, the embedding $\text{Pro}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \varprojlim \mathcal{A}$ is left s -filtering.*

(Proof) We use the fully exact embeddings $\text{Pro}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$ and $\varprojlim \mathcal{A} \hookrightarrow \text{Ind Pro}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$. The proof we give here is an analogue of the proof of Lemma 3.2 given in [10], where the embeddings $\mathcal{A} \hookrightarrow \text{Lex } \mathcal{A}$ and $\text{Ind}_{\mathbb{N}}^a \mathcal{A} \hookrightarrow \text{Ind } \mathcal{A} \hookrightarrow \text{Lex } \mathcal{A}$ are used.

We start by checking condition (3) of left s -filtering. Let $X = (\cdots \hookrightarrow X_{\bullet,-1} \hookrightarrow X_{\bullet,0} \hookrightarrow X_{\bullet,1} = X_{\bullet,1} = \cdots) \rightarrow Y = (\cdots \hookrightarrow Y_{\bullet,-1} \hookrightarrow Y_{\bullet,0} \hookrightarrow Y_{\bullet,1} \hookrightarrow Y_{\bullet,2} \hookrightarrow \cdots)$ be a map in $\varprojlim \mathcal{A}$, with $X = X_{\bullet,1}, Y_{\bullet,j} \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$. In $\text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$, the map consists of abelian group maps $\varinjlim_j \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X_{\bullet,j}) = \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X) \rightarrow \varinjlim_j \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j})$, $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$. Suppose $\text{id}_X \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(X, X)$ to be sent to $v \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(X, Y_{\bullet,j})$ for some j . Then for every $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$, a map $f \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X)$ is sent to $v \circ f$ by naturality, which lies in $\text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j})$ for the same j . Hence there is a j such that for every $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ the map $\text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X) \rightarrow \varinjlim_j \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j})$ factors through a map $\text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X) \rightarrow \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j})$. This means that the map $X \rightarrow Y$ factors over $Y_{\bullet,j}$ for that j . Since $0 \rightarrow \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j}) \hookrightarrow \varinjlim_{k \geq j} \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,k}) \twoheadrightarrow \varinjlim_{k \geq j} \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,k}/Y_{\bullet,j}) \rightarrow 0$ is a short exact sequence of abelian groups for every $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$, we have an admissible short exact sequence $Y_{\bullet,j} \hookrightarrow Y \twoheadrightarrow (0 \hookrightarrow Y_{\bullet,j+1}/Y_{\bullet,j} \hookrightarrow Y_{\bullet,j+2}/Y_{\bullet,j} \hookrightarrow \cdots)$ in $\text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$. We thus get a factorization $X \rightarrow Y_{\bullet,j} \hookrightarrow Y$ of $X \rightarrow Y$ with $Y_{\bullet,j} \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ and $Y_{\bullet,j} \hookrightarrow Y$ an admissible monomorphism.

To prove (4), let $Y \twoheadrightarrow X$ be an admissible epimorphism. As it is an epimorphism in $\text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$, we have a surjection $\varinjlim_j \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j}) \rightarrow \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X)$ for every $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$. There is a j such that for every $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ the map $\text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j}) \rightarrow \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X)$ is surjective. Indeed, there is a map $\mu \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(X, Y_{\bullet,j})$ for some j which is sent to $\text{id}_X \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(X, X)$. For every $f \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, X)$, the composition $\mu \circ f \in \text{Hom}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}}(Z, Y_{\bullet,j})$ is sent to f by naturality. Hence $Y_{\bullet,j} \twoheadrightarrow X$ is an epimorphism in $\text{Lex Pro}_{\mathbb{N}}^a \mathcal{A}$. In general, an idempotent complete exact category is closed under kernels of surjections in its abelian envelope. Noticing that $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ is idempotent complete by Proposition 2.3, we see that $Y_{\bullet,j} \twoheadrightarrow X$ is an admissible epimorphism in $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$. This means that the composition of $Y \twoheadrightarrow X$ with the admissible monomorphism $Y_{\bullet,j} \hookrightarrow Y$ is an admissible epimorphism, and thus we are done for (4).

Condition (1) follows from (3). Indeed, an admissible epimorphism $X \twoheadrightarrow Y$ factors through some $Z \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ such that $Z \hookrightarrow Y$ is an admissible monomorphism. The composition $X \twoheadrightarrow$

$Y \twoheadrightarrow Y/Z$ is 0, but since this composition is also an admissible epimorphism, Y/Z must be 0. This implies $Y \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$.

Finally, if $Y \hookrightarrow X$ is an admissible monomorphism, the quotient X/Y is in $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ by (1). As mentioned above, $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ is closed under kernels of surjections. Thus $Y = \ker(X \twoheadrightarrow X/Y)$ is in $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$, and condition (2) is proved. ■

Lemma 3.4 *There is a canonical null-homotopy of $K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})$.*

(Proof) Consider the functor $T : \text{Pro}_{\mathbb{N}}^a \mathcal{A} \rightarrow \text{Pro}_{\mathbb{N}}^a \mathcal{A}$ that sends a pro-object $X \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ of the form $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$ to the pro-object $T(X) \in \text{ob Pro}_{\mathbb{N}}^a \mathcal{A}$ given by $0 \leftarrow X_1 \leftarrow X_1 \oplus X_2 \leftarrow X_1 \oplus X_2 \oplus X_3 \leftarrow \cdots$. There is an isomorphism of exact functors $T \oplus \text{id}_{\text{Pro}_{\mathbb{N}}^a \mathcal{A}} \xrightarrow{\sim} T$, as $(T(X) \oplus X)_i \xrightarrow{\sim} T(X)_{i+1}$ defines an isomorphism of pro-objects. We write $K(T) + \text{id}_{K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})}$ for the sum of the maps $K(T)$ and $\text{id}_{K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})}$ defined by the direct sum in $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$, and $K(T) * \text{id}_{K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})}$ for the loop concatenation on $K(\text{Pro}_{\mathbb{N}}^a \mathcal{A}) = \Omega S(\text{Pro}_{\mathbb{N}}^a \mathcal{A})$. In general, the two operations $+$ and $*$ on a looped H -space produce homotopic maps. Hence the argument above shows that there is a homotopy between $K(T)$ and $K(T) * \text{id}_{K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})}$. By adding the inverse loop of $K(T)$ we obtain the desired canonical null-homotopy of $\text{id}_{K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})}$. ■

We remark that this is the argument used to prove that $K(\text{Ind}_{\mathbb{N}}^a \mathcal{A})$ is contractible ([10], Lemma 3.2).

Lemma 3.5 *There is an equivalence $\text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A} \xrightarrow[\longleftrightarrow]{\sim} \lim_{\leftarrow} \mathcal{A} / \text{Pro}_{\mathbb{N}}^a \mathcal{A}$.*

(Proof) We have a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \text{Ind}_{\mathbb{N}}^a \mathcal{A} \\ \downarrow & & \downarrow \\ \text{Pro}_{\mathbb{N}}^a \mathcal{A} & \longrightarrow & \lim_{\leftarrow} \mathcal{A}, \end{array}$$

whence there results a functor $F : \text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A} \rightarrow \lim_{\leftarrow} \mathcal{A} / \text{Pro}_{\mathbb{N}}^a \mathcal{A}$.

To construct a quasi inverse, we start by noticing that the functor $\text{Fun}^a(\Pi, \mathcal{A}) \rightarrow \text{Ind}_{\mathbb{N}}^a \mathcal{A}$, $(X_{i,j})_{i \leq j} \mapsto X_{0,1} \hookrightarrow X_{0,2} \hookrightarrow \cdots$, induces a functor $\tilde{G} : \lim_{\leftarrow} \mathcal{A} \rightarrow \text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A}$. Indeed, if $\phi \leq \psi : \mathbb{Z} \rightarrow \mathbb{Z}$ are bicofinal, the map $u_{X,\phi,\psi} : X \circ \tilde{\phi} \rightarrow X \circ \tilde{\psi}$ in $\text{Fun}^a(\Pi, \mathcal{A})$ is sent to the map $X_{\phi(0),\phi(\bullet)} \rightarrow X_{\psi(0),\psi(\bullet)}$, which factors as $X_{\phi(0),\phi(\bullet)} \hookrightarrow X_{\phi(0),\psi(\bullet)} \twoheadrightarrow X_{\psi(0),\psi(\bullet)}$. The map $X_{\phi(0),\phi(\bullet)} \hookrightarrow X_{\phi(0),\psi(\bullet)}$ is an isomorphism in $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ since it consists of natural isomorphisms $\varinjlim_j \text{Hom}_{\mathcal{A}}(A, X_{\phi(0),\phi(j)}) \xrightarrow{\sim} \varinjlim_j \text{Hom}_{\mathcal{A}}(A, X_{\phi(0),\psi(j)})$, $A \in \text{ob } \mathcal{A}$, as ϕ and ψ are bicofinal. We also see that $X_{\phi(0),\psi(\bullet)} \twoheadrightarrow X_{\psi(0),\psi(\bullet)}$ is a weak isomorphism in $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ with respect to \mathcal{A} , since it has the constant kernel $X_{\phi(0),\psi(0)} = X_{\psi(0),\psi(0)} = \cdots$. The functor \tilde{G} thus defined takes weak isomorphisms in $\lim_{\leftarrow} \mathcal{A}$ with respect to $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ to weak isomorphisms in $\text{Ind}_{\mathbb{N}}^a \mathcal{A}$ with respect to \mathcal{A} , since if $X \in \text{ob } \lim_{\leftarrow} \mathcal{A}$ is in the image of $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$ then its 0-th row is constant $X_{0,1} = X_{0,1} = \cdots$, i.e. $\tilde{G}(X)$ is in the image of \mathcal{A} . Hence \tilde{G} factors through a functor $G : \lim_{\leftarrow} \mathcal{A} / \text{Pro}_{\mathbb{N}}^a \mathcal{A} \rightarrow \text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A}$.

We have $G \circ F = \text{id}_{\text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A}}$ by definition. On the other hand, if $X = (X_{i,j})_{i \leq j} \in \text{ob} \lim_{\leftarrow} \mathcal{A}$, then $F \circ G(X)$ is the object \tilde{X} of $\lim_{\leftarrow} \mathcal{A}$ determined by $\tilde{X}_{i,j} = \tilde{X}_{0,j} = X_{0,j}$, ($i \leq 0 < j$). Define an admissible epimorphism $f_X : X \rightarrow \tilde{X}$ in $\text{Fun}^a(\Pi, \mathcal{A})$ (hence in $\lim_{\leftarrow} \mathcal{A}$) by

$$(f_X)_{i,j} = \begin{cases} X_{i,j} = X_{i,j} & (0 \leq i \leq j) \\ X_{i,j} \twoheadrightarrow X_{0,j} & (i \leq 0 < j) \\ X_{i,j} \twoheadrightarrow 0 & (i \leq j \leq 0). \end{cases}$$

The kernel coincides with the image of $0 \leftarrow X_{-1,0} \leftarrow X_{-2,0} \leftarrow X_{-3,0} \leftarrow \cdots \in \text{ob} \text{Pro}_{\mathbb{N}}^a \mathcal{A}$ in $\lim_{\leftarrow} \mathcal{A}$. Hence f_X is a weak isomorphism in $\lim_{\leftarrow} \mathcal{A}$ with respect to $\text{Pro}_{\mathbb{N}}^a \mathcal{A}$. Thus we get an isomorphism $f : \text{id}_{\lim_{\leftarrow} \mathcal{A} / \text{Pro}_{\mathbb{N}}^a \mathcal{A}} \xrightarrow{\sim} F \circ G$, to conclude that G is a quasi inverse to F . \blacksquare

We now have, if \mathcal{A} is idempotent complete, a commutative diagram of K -theory spaces

$$\begin{array}{ccccc} K(\mathcal{A}) & \longrightarrow & K(\text{Ind}_{\mathbb{N}}^a \mathcal{A}) & \longrightarrow & K(\text{Ind}_{\mathbb{N}}^a \mathcal{A} / \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ K(\text{Pro}_{\mathbb{N}}^a \mathcal{A}) & \longrightarrow & K(\lim_{\leftarrow} \mathcal{A}) & \longrightarrow & K(\lim_{\leftarrow} \mathcal{A} / \text{Pro}_{\mathbb{N}}^a \mathcal{A}), \end{array}$$

where the horizontal sequences are homotopy fibrations by Theorem 3.1, Lemmas 3.2 and 3.3. Moreover, the third vertical map is an equivalence by Lemma 3.5. It follows that the square

$$\begin{array}{ccc} K(\mathcal{A}) & \longrightarrow & K(\text{Ind}_{\mathbb{N}}^a \mathcal{A}) \\ \downarrow & & \downarrow \\ K(\text{Pro}_{\mathbb{N}}^a \mathcal{A}) & \longrightarrow & K(\lim_{\leftarrow} \mathcal{A}) \end{array}$$

is homotopy cartesian. As pointed out above (Lemmas 3.2 and 3.4), there are canonical null-homotopies of $K(\text{Ind}_{\mathbb{N}}^a \mathcal{A})$ and $K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})$. Hence there results a homotopy equivalence $K(\mathcal{A}) = \text{holim}(K(\text{Ind}_{\mathbb{N}}^a \mathcal{A}) \rightarrow K(\lim_{\leftarrow} \mathcal{A}) \leftarrow K(\text{Pro}_{\mathbb{N}}^a \mathcal{A})) \xrightarrow{\sim} \text{holim}(* \rightarrow K(\lim_{\leftarrow} \mathcal{A}) \leftarrow *) = \Omega K(\lim_{\leftarrow} \mathcal{A})$, and the proof of Theorem 1.2 is complete.

3.1 Delooping the Waldhausen spaces

We can also prove the original statement of Conjecture 1.1, that the Waldhausen space $S(\lim_{\leftarrow} \mathcal{A})$ is a delooping of $S(\mathcal{A})$, under the assumption that \mathcal{A} is idempotent complete.

Theorem 3.1 remains valid when the involved K -theory spaces are replaced by the Waldhausen spaces. The proof of Theorem 3.1 in [10] uses the homotopy fibration sequence $wS_{\bullet} \mathcal{W} \rightarrow wS_{\bullet} S_{\bullet}(f) \rightarrow wS_{\bullet} S_{\bullet} \mathcal{V}$ constructed by Waldhausen for an exact functor $f : \mathcal{V} \rightarrow \mathcal{W}$ of Waldhausen categories ([12], 1.5.5). In the case where the ambient categories are exact and the weak equivalences are the isomorphisms, Schlichting [10] showed that if f is the left or right s -filtering inclusion $\mathcal{A} \hookrightarrow \mathcal{U}$ of the idempotent complete exact category \mathcal{A} , then the map $\mathcal{U} \rightarrow \mathcal{U} / \mathcal{A}$ induces a homotopy equivalence $iS_{\bullet} S_{\bullet}(\mathcal{A} \hookrightarrow \mathcal{U}) \xrightarrow{\sim} iS_{\bullet} S_{\bullet}(0 \hookrightarrow \mathcal{U} / \mathcal{A})$, whose target is in turn equivalent

to $iS_\bullet \mathcal{U}/\mathcal{A}$ ([10], 2.3). Hence there results a fibration sequence $iS_\bullet \mathcal{A} \rightarrow iS_\bullet \mathcal{U} \rightarrow iS_\bullet \mathcal{U}/\mathcal{A}$. The fibration sequence $K(\mathcal{A}) \rightarrow K(\mathcal{U}) \rightarrow K(\mathcal{U}/\mathcal{A})$ is obtained by taking the realization, but we notice that the argument above in fact proves that there is a fibration sequence of Waldhausen spaces $S(\mathcal{A}) \rightarrow S(\mathcal{U}) \rightarrow S(\mathcal{U}/\mathcal{A})$.

The space $S(\text{Ind}_{\mathbb{N}}^a \mathcal{A})$ is contractible since $\pi_0(S(\text{Ind}_{\mathbb{N}}^a \mathcal{A})) = 0$, as the Waldhausen space of a category is always connected, and we also have $\pi_{i+1}(S(\text{Ind}_{\mathbb{N}}^a \mathcal{A})) = \pi_i(K(\text{Ind}_{\mathbb{N}}^a \mathcal{A})) = 0$ for $i \geq 0$ by Lemma 3.2. The homotopy fibration sequence $S(\mathcal{A}) \rightarrow S(\text{Ind}_{\mathbb{N}}^a \mathcal{A}) \rightarrow S(\text{Ind}_{\mathbb{N}}^a \mathcal{A}/\mathcal{A})$, whose second term is contractible, gives a homotopy equivalence between $S(\mathcal{A})$ and $\Omega S(\text{Ind}_{\mathbb{N}}^a \mathcal{A}/\mathcal{A})$. On the other hand, the space $S(\text{Pro}_{\mathbb{N}}^a \mathcal{A})$ is also contractible by Lemma 3.4. We obtain a homotopy fibration sequence $S(\text{Pro}_{\mathbb{N}}^a \mathcal{A}) \rightarrow S(\varprojlim \mathcal{A}) \rightarrow S(\varprojlim \mathcal{A}/\text{Pro}_{\mathbb{N}}^a \mathcal{A})$, whose first term is contractible, and third term equivalent to $S(\text{Ind}_{\mathbb{N}}^a \mathcal{A}/\mathcal{A})$ by Lemma 3.5.

By summing up these we get a homotopy equivalence between $S(\mathcal{A})$ and $\Omega S(\varprojlim \mathcal{A})$.

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